Calculation of tunneling lifetimes in WKB approximation

Review of WKB

\[ V(x) \]

To solve a 1 dim. T. Eqn.

\[ -\frac{h^2}{2m} \psi''(x) + V(x) \psi(x) = E \psi(x), \]

introduce \( k^2(x) = \left[ \frac{2m}{E-V(x)} \right] \frac{1}{\hbar^2} \)

\[ \Rightarrow \psi''(x) + k^2(x) \psi(x) = 0 \]

and observe that if \( k(x) \) varies "slowly" with \( x \), the approximate solution should look like:

\[ \psi(x) \approx \exp \left( i \int k(x') \, dx' \right) \]

And a careful derivation gives the general solution as

\[ \psi(x) \approx C_+ \frac{e^{+ \int k(x') \, dx'}}{\sqrt{k(x)}} + C_- \frac{e^{- \int k(x') \, dx'}}{\sqrt{k(x)}} \]

in any "classically allowed" region where \( E > V(x) \), but instead

\[ \psi(x) \approx d_+ \frac{e^{+ \int k(x') \, dx'}}{\sqrt{k(x)}} + d_- \frac{e^{- \int k(x') \, dx'}}{\sqrt{k(x)}} \]

where \( k(x) = \left[ \frac{2m}{\hbar^2} (V(x) - E) \right] \frac{1}{\hbar^2} \)

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The approximate solutions first written blow up and fail for $x$ close to any classical turning point since $k(x) \to 0$ or $k'(x) \to 0$.

In fact, accurate solution of the Sch. Eqn. shows that an oscillatory solution to the left of a turning point like the one at $x = b$ evolves into a specific exponentially rising/falling solution to the right.

(For a derivation, see, e.g., Mezibacher's QM text.)

\[
\begin{align*}
\frac{2}{\sqrt{k(x)}} \cos \left( \int_{x}^{b} k(x') \, dx' - \frac{\pi}{4} \right) & \quad \rightarrow \quad \frac{1}{k(x)} e^{-\int_{b}^{x} k(x') \, dx'} + \int_{b}^{x} k(x') \, dx' \\
\frac{1}{\sqrt{k(x)}} \cos \left( \int_{x}^{b} k(x') \, dx' + \frac{\pi}{4} \right) & \quad \rightarrow \quad \frac{1}{k'(x)} e^{-\sin \left( \int_{x}^{b} k(x') - \frac{\pi}{4} \right)}
\end{align*}
\]
whereas if the class, allowed region is to the **RIGHT** we have instead

\[
\psi(x) = \frac{1}{\sqrt{k(x)}} e^{-i \int_b^x k(x') dx'} = \frac{1}{\sqrt{k(x)}} e^{-i \int_b^x k(x') dx'} + \frac{i}{\sqrt{k(x)}} \sin \left( \int_b^x k(x') dx' - \frac{\pi}{4} \right)
\]

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**Tunneling Probability through a finite barrier**

Let's assume that \( V(x) \to 0 \) at \( x \to \pm \infty \)

\( \Rightarrow \) The solution appropriate to scattering of a particle incident from the left should have only a transmitted wave on the right, i.e. in WKB approx

\[
\psi(x) \approx \frac{1}{\sqrt{k(x)}} e^{i \int_a^x k(x') dx' - \frac{\pi}{4}} \quad \text{(} \frac{\pi}{4} \text{ added just for convenience)}
\]

at \( x \gg c \)

\[
= \frac{1}{\sqrt{k(x)}} \cos \left( \int_c^x k(x') dx' - \frac{\pi}{4} \right) + \frac{i}{\sqrt{k(x)}} \sin \left( \int_c^x k(x') dx' - \frac{\pi}{4} \right)
\]
\[ S_x^c = S_x^c + S_x^b = S_x^c - S_x^b \]

and by the connection formulas we can make the region II solution as

\[ \psi_\Pi(x) = \frac{1}{2\sqrt{k(x)}} e^{-i} - \frac{1}{\sqrt{k(x')}} \]

which is valid for \( b < x < c \).

And another application of connection formulas gives the region I solution as follows. First rewrite \( \psi_\Pi(x) \):

\[ \psi_\Pi(x) = \frac{1}{2\sqrt{k(x)}} \int_{b}^{x} k(x') dx' - \frac{i}{\sqrt{k(x')}} \int_{b}^{x} k(x') dx' \]

and now using the first set of connection formulas,

\[ \psi_\Pi(x) = -\frac{1}{2} \frac{1}{4i} \sin \left( \int_{b}^{x} k(x') dx' - \frac{\pi}{4} \right) - \frac{2}{\sqrt{k(x')}} \cos \left( \int_{b}^{x} k(x') dx' - \frac{\pi}{4} \right) \]

\[ = \frac{1}{\sqrt{k(x)}} \left\{ \frac{-\pi}{4i} + \frac{1}{\pi} \right\} \text{ reflected wave} \]

\[ - \frac{1}{\sqrt{k(x')}} \left\{ \frac{\pi}{4i} + \frac{1}{\pi} \right\} \text{ incident wave} \]

Thus the transmission probability per collision is

\[ P_{\text{trans}} = \left| \frac{-\pi}{4i} + \frac{1}{\pi} \right|^2 = \frac{\left| \pi \right|^2}{1 + \frac{\pi^2}{4}} \sim \left| \frac{\pi}{4} \right|^2 \]

or

\[ P_{\text{trans}} = e^{-2 \int_{b}^{c} k(x') dx'} \]
Classically, a particle bound between $x=a$ and $x=b$ simply bounces back and forth with a period $\Delta t_{\text{class}} = 2\int_a^b \frac{dx}{v(x)}$ where $v(x)$ is the local velocity $\sqrt{\frac{2(E-V(x))}{m}}$.

The loss of probability due to tunneling is equal to

$$\frac{\Delta P(t)}{\Delta t} = \frac{|T|^2}{\Delta t_{\text{class}}} P(t)$$

or

$$\frac{dP}{dt} = -\Theta(t) e^{\frac{-2\int_b^a H(x') dx'}{2\int_a^b dx' \left( \frac{m}{2(E-V(x'))} \right)^{1/2}}}$$

$$\Rightarrow P(t) = e^{-\Gamma t}$$

where

$$\Gamma = \frac{1}{T_{\text{tunnel}}} = \frac{e^{-2\int_b^a H(x') dx'}}{\int_a^b \left( \frac{2m}{E-V(x')} \right)^{1/2} dx'}$$