Notation. We introduce a complex 

$$X = \{ K_1, K_2, \ldots, K_n \}$$

as a collection of group elements without respect to order.

(i) Can multiply a complex by an element, e.g.,

$$KX = \{ K_1X, K_2X, \ldots, K_nX \}$$

where $X$ is any group element.

(ii) Can multiply a complex by a complex

$$\Rightarrow K'R' = \{ K_1R_1, K_2R_2, \ldots, K_nR_n, K'_1R'_1, K'_2R'_2, \ldots \}$$

Again, a collection of elements $C$ is a class if $X^{-1}C X = C$ for $X$ in $H$.

Consider the product of $2$ classes $C_i C_j$

$$\Rightarrow C_i C_j = (X^{-1}C_i X)(X^{-1}C_j X)$$

$$= X^{-1}(C_i C_j) X$$

$$\Rightarrow C_i C_j$$ consists of complete classes.

i.e., we can write the product $C_i C_j$ symbolically as a "sum" of single but complete classes, as

$$C_i C_j = \sum_{k} a_{ij,k} C_k$$

where $a_{ij,k}$ is integer telling how many times the complete class $C_k$ appears.
Equilateral Triangle Example

There are 3 classes:

\[ C_1 = \{E\} \]
\[ C_2 = \{A, B, C\} \]
\[ C_3 = \{D, F\} \]

\[ C_1 C_1 = C_1 \quad C_1 C_2 = C_2 \quad C_1 C_3 = C_3 \]

\[ C_2 C_2 = \{AA, AB, AC, BA, BB, BC, CA, CB, CC\} \]
\[ = \{E, D, F, F, E, D, D, F, E\} \]
\[ = 3C_1 + 3C_3 \]

\[ C_2 C_3 = \{AD, AF, BD, BF, CD, CF\} \]
\[ = \{B, C, A, C, A, B\} \]
\[ = 2C_2 \]

\[ C_3 C_3 = \{D^2, DF, E^2, ED\} = \{F, E, E, D\} \]
\[ = 2C_1 + C_3 \]

**Def.** We saw earlier that there are a finite number, 
(l - 1 where \(l = \frac{6}{2}\)) of distinct cosets for any subgroup \(G\). Call each of these a complex \(K_i\). Then if \(G\) is an invariant subgroup, we know that \(G K_i = K_i G\) i.e. \(K_i^{-1} G K_i = G\).

Moreover \(G K_i = G K_j\) i.e. \(K_i \cap K_j\) are in the same coset. And this set of \((l - 1)\) complexes plus \(G = \{G, K_2 \cdots K_l\}\) is a smaller group itself called the Factor Group of \(H\) w.r.t. the normal divisor (inv. subgroup) \(G\). (And "E" = G here)
Representation Theory  (Tinkham, Chap. 3)

**GOAL** To develop the method of representing elements of a group by matrices, with matrix multiplication for group multiplication

**Def** Let $G = \{E, A, B, C, \ldots \}$ be a finite group of order $g$ with $E = \text{identity}$

$\Rightarrow$ Let $T = \{ T(E), T(A), \ldots \}$ be a set of nonsingular matrices of the same dimension, such that $T(A) T(B) = T(AB)$

i.e. if $AB = C$ in group $G$, then $T(A) T(B) = T(C)$ in $T$

Then the set $T$ of matrices is called a **representation** of group $G$.

and the # rows (or columns) = dimension of the representation
Two possible cases

1. If all matrices of the set \( T \) are distinct, i.e. there is no \( T(X) = T(Y) \), then
   a) there is a one-to-one correspondence between the elements of \( G \) and the elements/matrices of \( T \), i.e. for each element of \( G \) there corresponds exactly one element of \( T \).
   b) Groups \( G \) and \( T \) are isomorphic.
   c) Then the representation of \( G \) by matrices \( T \) is called a \underline{faithful} representation.

2. If the matrices are \underline{not all distinct},
   \( \Rightarrow \) there is only a \underline{homomorphism} from \( G \) to \( T \) and the representation is \underline{unfaithful}.

Notice that a trivial representation of any group is \( T(e) = I \), \( T(A) = I \), ...

This is called the identity representation, and it is unfaithful, for \( g \neq 1 \).

Note: One can prove that every group has at least one faithful (i.e. physically useful) representation.
Example from QM

Let the elements of $G$ represent operators in an $n$-dimensional vector space, and let $\{ \phi_i \}$ be a complete, orthonormal basis.

$$\Rightarrow A \phi_i = \sum_j \phi_j T_{ji}(A)$$

i.e. $A|\phi_i\rangle = \sum_j |\phi_j\rangle \langle \phi_j| A|\phi_i\rangle$

This logic helps to remember the order, why it is $\phi_j T_{ji}(A)$

$$\Rightarrow$$ These matrices $T(E), T(A)$... generate a representation of $G$ since

$$\hat{A}(\hat{B} \phi_i) = \hat{A} \left( \sum_j \phi_j T_{ji}(B) \right)$$

$$= \sum_j \sum_k \phi_k T_{kj}(A) T_{ji}(B)$$

or $\hat{A}\hat{B} \phi_i = \sum_k \phi_k T_{ki}(AB)$

these two expressions agree provided

$$\sum_j T_{kj}(A) T_{ji}(B) = T_{ki}(AB)$$

i.e. $T(A) T(B) = T(AB)$
Properties of group representations

1) The matrix corresponding to $E$ must be the identity matrix, $T(E) = I = E$

2) $T(A^{-1}) = [T(A)]^{-1}$

3) Suppose we have 2 representations of a group $G$, i.e.

$T_1 = \{ T_1(E), T_1(A), T_1(B), \ldots \}$

$T_2 = \{ T_2(E), T_2(A), T_2(B), \ldots \}$

If there exists a nonsingular matrix $S$, such that

$T_1(A) = S^{-1} T_2(A) S$

$T_1(B) = S^{-1} T_2(B) S$

i.e. with the same matrix $S$ for every group element, they are said to be EQUIVALENT REPRESENTATIONS.

If no $S$ exists such that this is true, then $T_1$ and $T_2$ are said to be INEQUIVALENT or DISTINCT.
Reducible vs. Irreducible

Recall— we saw earlier how we can always take 2 or more representations of a group, and make a new (faithful) representation by combining the matrices into larger matrices.

\[
T(A) = \begin{pmatrix}
T_1(A)_{n \times n} & 0 \\
0 & T_2(A)_{m \times m}
\end{pmatrix}
\]

artificially enlarged to \((n+m) \times (n+m)\)

Such representations are said to be REDUCIBLE.

The following discussion quantities this.

* Note first of all that reducibility of a given representation can be concealed by the application of a similarity transformation, \(S\), which mixes up rows, columns, etc.

More general definition: If all matrices in a given representation can be transformed into block-diagonal form by the same similarity transform, \(S\), then the representation is called REDUCIBLE.

If no such matrix \(S\) exists such that all \(T(X)\) can be transformed to block-diagonal form, then it is called an IRREDUCIBLE REPRES. or IRREP.
Notation. When a reducible repr. $T$ is put into its block-diagonal form, with each block $T^{(1)}, T^{(2)}, \ldots$ irreducible, we indicate the breakdown of $T$ as

$$T = T^{(1)} + 2T^{(2)} + \ldots$$

(this does not mean an actual sum)

or more generally

$$T = \sum_{i} a_i T^{(i)}$$

When this has been accomplished, we call it a decomposition into its irreducible parts.
For instance, there turn out to be 3 irreducible representations for the equilateral triangle group:

\[
\begin{array}{ccccccc}
\text{Dimensionality} & E & A & B & C & D & F \\
\hline
\lambda_i = 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\lambda_i = 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
\lambda_i = 2 & \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, & \begin{pmatrix}
\frac{1}{2} & \frac{-\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & \frac{1}{2}
\end{pmatrix}, & \begin{pmatrix}
\frac{1}{2} & \frac{-\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & \frac{1}{2}
\end{pmatrix}, & \begin{pmatrix}
\frac{1}{2} & \frac{-\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & \frac{-1}{2}
\end{pmatrix}, & \begin{pmatrix}
\frac{1}{2} & \frac{-\sqrt{3}}{2} \\
\frac{-\sqrt{3}}{2} & \frac{-1}{2}
\end{pmatrix}
\end{array}
\]

Note: \( \text{Tr}\{E^2\} = 2 \) for the \( \lambda_i = 2 \) irrep

\[
\text{Tr}\{A^2\} = 0 = \text{Tr}\{B\} = \text{Tr}\{C\} \\
\text{Tr}\{D\} = -1 = \text{Tr}\{F\}
\]

\( \Rightarrow \) Traces are in fact the same for all matrices in the same class, as we had stated previously.
Lemma 1  Equivalence to a unitary representation

Consider the following matrix, which is Hermitian, by construction:

\[ H = \sum_{i=1}^{h} A_i A_i^\dagger \]

call \( T(A_i) \rightarrow A_i \) for each of the \( h \) group elements.

Since \( H \) is Hermitian, it can be diagonalized by a unitary transformation \( U \)

\[ U^\dagger H U = \text{diagonal} \]

eigenvalues which here must be real and positive.

Then,

\[ d = \sum_i (U^\dagger A_i U)(U^\dagger A_i U) = \sum_i A_i A_i^\dagger \]

where \( U^\dagger A_i U = A_i^\dagger \), \( U^\dagger A_i^\dagger U = A_i \)

\[ \Rightarrow E = d^{-\frac{1}{2}} \sum_i A_i A_i^\dagger d^{-\frac{1}{2}} \]

Then define \( A_j = d^{-\frac{1}{2}} A_j d^{-\frac{1}{2}} \)

and we have each \( A_j \) will now be unitary.
To see this, note

\[ A_j A_j^* + = \left( d^{-\frac{1}{2}} A_j \ d^2 \right) E \left( d^{\frac{1}{2}} A_j^* \ d^{-\frac{1}{2}} \right) \]

\[ = \left( d^{-\frac{1}{2}} A_j \ d^{\frac{1}{2}} \right) \left( d^{-\frac{1}{2}} \sum_k A_k A_k^* \ d^{\frac{1}{2}} \right) \left( d^{\frac{1}{2}} A_j^* \ d^{-\frac{1}{2}} \right) \]

\[ = d^{-\frac{1}{2}} \sum_k (A_j A_k^*) (A_j A_k^*)^+ d^{-\frac{1}{2}} \]

This is in the group, \( T \) by the rearrangement theorem, every element appears exactly once, whereby

\[ A_j A_j^* + = d^{-\frac{1}{2}} \sum_i A_i A_i^* d^{-\frac{1}{2}} = E \]

So starting from an arbitrary repr., this procedure shows how to make all of the repr. matrices unitary.
Schur's Lemma: Any matrix that commutes with all matrices of an irreducible representation must be a constant matrix.

- i.e., if a nonconstant commuting matrix exists for some representation, it must be reducible.

(A "constant matrix" means a constant scalar multiple of the identity)

Proof: Let $M$ be a matrix that commutes with all matrices of a representation (unitary, by lemma).

$$
A_i \cdot M = M \cdot A_i, \quad i = 1, \ldots, h
$$

then

$$
M^+ A_i^+ = A_i^+ M^+
$$

and of course $A_i^{-1} M = M A_i^{-1}$ since $A_i^{-1}$ is in the group.

$$
A_i \cdot (M^+ A_i^+) A_i = A_i \cdot (A_i^+ M^+) A_i
$$

$$
\Rightarrow A_i M^+ = M^+ A_i
$$

and $M A_i^+ = A_i^+ M$

we want to prove that $M = c \cdot I = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

Now, if $M$ and $M^+$ commute, then so do the Hermitian matrices

$$
H_1 = M + M^+
$$

$$
H_2 = i \cdot (M - M^+)
$$

So, continue this theorem using the Hermitian matrix $H$, without loss of generality.
Now, begin by diagonalizing $H$:

\[ d = U H U = \text{diagonal} \]

i.e. \[ d = U \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U^+ \]

and now perform a similarity transformation on each $A_i$, defining

\[ A_i' = U^{-1} A_i U = \begin{pmatrix} \lambda_1 A_i \\ \vdots \\ \lambda_n A_i \end{pmatrix} U^+ \]

\[ \Rightarrow H A_i = A_i H \]

\[ U^+ d U^+ A_i = A_i U d U^+ \]

\[ \Rightarrow d (U^+ A_i U) = (U^+ A_i U) d \]

or \[ d A_i = A_i' d \] (d commutes with $A_i$)

explicitly, this means that \[ d_{\mu\nu} (A_i')_{\nu\lambda} = (A_i')_{\mu\lambda} d_{\nu\nu} \]

whereby

\[ (A_i')_{\mu\nu} (d_{\mu\nu} - d_{\lambda\nu}) = 0 \]

This is satisfied if either (i)

\[ \frac{d_{\mu\nu}}{d_{\mu\nu}} = d_{\lambda\nu} = C \text{ for all } \mu, \nu \]

i.e. $H = cE$

Since then $d = U (cE) U$

\[ = cE \]

OR (ii) \[ (A_i')_{\mu\nu} = 0 \text{ for all } \mu, \nu \text{ having } d_{\mu\nu} \neq d_{\lambda\nu} \]

i.e. \[ d = \begin{pmatrix} d_{\mu\nu} & 0 & \cdots & 0 \\ 0 & d_{\mu\nu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{\mu\nu} \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \]
meaning that \((A_i')_{\mu
u} = \begin{pmatrix} (A_i')_{\mu
u} & 0 \\ 0 & (A_i')_{\nu\nu} \end{pmatrix}\) 

\(\implies A_i'\) is a reducible representation.

So in summary, if the representation was IRREDUCIBLE, then \(M = cE\).

**Lemma 3** Suppose we are given 2 irreducible representations of a group \(H\),

\(T^{(1)}(A_i)\) of dimensionality \(l_1\),

and \(T^{(2)}(A_i)\) of dimensionality \(l_2\).

Now suppose a rectangular matrix \(M(l_2 \times l_1)\) exists such that

\[M T^{(1)}(A_i) = T^{(2)}(A_i) M, \quad i=1, \ldots, h\]

Then either (1) if \(l_1 \neq l_2\), \(M = 0\)

\(\) or (2) if \(l_1 = l_2\), \(M = 0\) or else \(\det M = 0\) (and if this is true, \(M^{-1}\) exists, consequently

\[M T^{(1)}(A_i)^{-1} M = T^{(2)}(A_i)\]

and the 2 representations are equivalent.
Proof of lemma 3 - this proof assumes unitary representations without loss of generality. And take \( \ell_1 \leq \ell_2 \) to be definite.

Start from \( MT^{(i)}(A_i) = T^{(2)}(A_i^*) M, \quad i = 1, \ldots, h \)
and \( T^{(i)}(A_i^*) M^+ = M^+ T^{(2)}(A_i^*) \)

or \( T^{(i)}(A_i^*) M^+ = M^+ T^{(2)}(A_i^*) \)

left-mult by \( M \)

and of course \( MT^{(i)}(A_i^*) = T^{(2)}(A_i^*) M \)

right-mult by \( M^+ \)

\[ \Rightarrow M M^+ T^{(2)}(A_i^*) = T^{(2)}(A_i^*) M M^+ \]

\[ \Rightarrow \text{We have found a matrix}, \; MM^+, \text{which commutes with all the matrices of this IRREP.} \]

\[ \Rightarrow MM^+ = cE, \text{ by Schur's lemma.} \]

Consequences Let \( \ell_1 = \ell_2 \) first, so that \( M \) = square matrix

\[ \Rightarrow |\det(M)|^2 = c \ell_1 \]

Hence a) if \( c \neq 0, \; \det(M) \neq 0 \) and \( M^{-1} \) exists whereby \( T^{(1)} \) and \( T^{(2)} \) are equiv. reps. and \( M \) = similarity-transf. matrix

or b) if \( c = 0 \Rightarrow MM^+ = 0 \Rightarrow M = 0 \)

\[ \sum_k |M_{ik}|^2 = (MM^+)_{ii} \]
and notice that $MM^+ = NN^+$
and $N$ clearly has $\det(N) = 0$

$\Rightarrow \det(NN^+) = \det(MM^+) = 0$

But we saw that $MM^+ = CE$
$\Rightarrow c = 0$ which can be true only if $M = 0$ itself! and we're done!
The next result is **AMAZING**:

The **GREAT ORTHOGONALITY THEOREM**

Consider any 2 inequivalent, irreducible, unitary representations of group \( \mathcal{H} \) of order \( h \), namely \( \tau^{(i)} \) and \( \tau^{(j)} \)

Then

\[
\sum_{R} \left[ \tau^{(i)}(R) \right]^*_{\mu \nu} \left[ \tau^{(j)}(R) \right]_{\alpha \beta} = \frac{h}{l_i} \delta_{ij} \delta_{\mu \alpha} \delta_{\nu \beta}
\]

where the sum over \( R \) includes ALL group elements \( E, A_2 \ldots A_h \),

and where \( l_i = \text{dimensionality of } \tau^{(i)} \)

**Proof**

First consider the case \( i \neq j \Rightarrow \) irreps inequivalent

Then we claim that a matrix obeying lemma 3 is

\[
M = \sum_{R} \tau^{(j)}(R) \times \tau^{(i)}(R^{-1})
\]

where \( \times = \text{arbitrary } l_2 \times l_1 \text{ matrix} \)

Verification

\[
\tau^{(j)}(A_i) M = \sum_{R} \tau^{(j)}(A_i;R) \times \tau^{(i)}(R) \times \tau^{(i)}(R^{-1})
\]

\[
\Rightarrow \tau^{(j)}(A_i) M = \sum_{R} \tau^{(j)}(A_i;R) \times \tau^{(i)}(R^{-1} A_i^{-1}) \tau^{(i)}(A_i)
\]

\[
= \sum_{R'} \tau^{(j)}(R') \times \tau^{(i)}(R'^{-1}) \tau^{(i)}(A_i)
\]

hence

\[
\tau^{(j)}(A_i) M = M \tau^{(i)}(A_i)
\]