Method of Steepest Descent (saddle point method) (stationary phase)

Consider a general function of \( z \), \( S(z) \), with a contour integral representation of the form:

\[
S(z) = \int g(t) e^{z f(t)} dt \quad \text{with } z, f(t) \text{ complex}
\]

(e.g., this form arises for Laplace, Fourier transforms) "smooth function"

We will assume that the integrand goes to zero at the ends of the contour.

Goal: Find an approximate expression for this integral, valid for \( S(z) \) at \( |z| \to \infty \)

Nature of \( e^{z f(t)} \):

\[
\text{Re}[z f(t)] \quad \text{Re}[z f(t)] \quad i \text{Im}[z f(t)]
\]

\[
\text{Since } e \cdot e = e \quad e
\]

\( e^{z f(t)} \) oscillates along the imaginary axis, but blows up along the real axis, and decays exponentially along the negative real axis.
For a general complex value of $z$, write

$$z = R (\cos \theta + i \sin \theta) = R e^{i \theta}$$

$$\Rightarrow \text{Im}[z + f(t)] = R \sin \theta f(t), \text{ if } f(t) \text{ is real}$$

So, qualitatively, in the function

$$\exp\left(i \text{Im}[z + f(t)]\right)$$

this can be seen as being the "frequency" of oscillation, and as $|z|$ increases, this function oscillate faster and faster.

plot

i.e., $\text{Re}[\exp(i \text{Im}(z + f(t)))]

$\text{larger } |z|$

Such fast oscillations are bad because they imply that there will be challenging cancellation effects when you perform a numerical integration.
Qualitative issues

We expect that the integral will be dominated by regions along \( C \) where \( \Re[z + f(z)] = \text{maximum} \).

THEOREM

However, a complex, analytic function can have no maxima or minima inside a finite region of the complex plane. Call \( t = x + iy \).

Why? Because \( \nabla^2 U(t) = 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \)

(where here \( U(t) = z^2(t) \)).

If we suppose that \( f'(t) = 0 \), then if we have a maximum of \( \frac{\partial}{\partial t} z^2(t) \) at \( t_0 \), when we approach \( t_0 \) along \( x \), \( u \) i.e., \( \frac{\partial^2 u}{\partial x^2} < 0 \).

Then in order for \( \nabla^2 u \) to be zero, to

\( \frac{\partial^2 u}{\partial y^2} > 0 \)

\( \Rightarrow \text{minimum} \)

when \( t_0 \) is approached along the \( y \)-axis!
And we have 2 basic needs for this contour:

1. \[ \text{Re} \left[ z + f(t) \right] \text{ is a local maximum along the contour path, i.e. } \]
   \[ z + f'(t) = 0 \text{ and we must find the critical point(s) where this is satisfied} \]

and

2. Want to keep \[ \text{Im} \left[ z + f(t) \right] = \text{constant} \] versus \[ t \]
   along the contour, near the point(s) to identified,
   \[ f'(t_0) = 0 \]
   \[ i \text{Im}(z + f(t)) + \text{Re}(z + f(t)) \]
   \[ \Rightarrow J(t) = \int_{c} g(t) e^{i \text{Im}(z + f(t))} dt \]

will be dominated by such regions where the oscillations get "slow" i.e. where the phase is approximately stationary!

Then, if \[ C \] passes through such a region near \[ t = t_0 \], we can approximately take \[ f(t) \] outside the integral since it is roughly constant, and also \[ g(t_0) \] since it is "smooth".

\[ \Rightarrow J(t) = g(t_0) \exp(i \text{Im}[z + f(t_0)]) \int_{c} \exp(i \text{Re}[z + f(t)]) dt \]

near \[ t_0 \]
So if \( \text{Re}(z + \epsilon t) \neq \text{maximum} \) at the ends of the contour, we want to search for \( \epsilon \) obeying

\[
2 f'(t_0) = 0 \quad \text{or} \quad f''(t_0) = 0 = \frac{df}{dt} |_{t_0}
\]

Once we have found such a \( \epsilon \), we can expand:

\[
f(t) = f(t_0) + f'(t_0)(t - t_0) + f''(t_0) \frac{(t - t_0)^2}{2} + O((t - t_0)^3)
\]

Then

\[
i \text{Im}[z + f(t)] \approx i \text{Im}(z + f(t_0))
\]

and

\[
i \text{Im}(z + f(t)) = \epsilon e^{i \text{Im}(z + f(t_0))}
\]

\[
\text{Re}(z + f(t)) = \frac{1}{2} \text{Re}(z + f(t_0)) + \epsilon e^{i \text{Im}(z + f(t_0))} \frac{1}{2} \text{Re}(z + f(t_0)) (t - t_0)^2 + O((t - t_0)^3)
\]
To calculate $\mathcal{I}(z) = \int_C g(t) e^{zt} \, dt$ at $|z| \to \infty$,
expand $f(t)$ about a critical point $t_0$
found by solving $f'(t_0) = 0$
$\Rightarrow f(t) \approx f(t_0) + \frac{1}{2} f''(t_0) (t - t_0)^2$, near $t_0$

Next, deform contour $C$ to pass through $t_0$
at an angle $\alpha$ to be determined by requiring
the integrand to decrease as steeply as possible
(steepest descent). That is, set
\[ t - t_0 = \tau e^{i\phi} \quad \text{with} \quad \alpha = \text{constant} \quad \tau \in \text{real} \]

To find $\alpha$, set $z = |z| e^{i\phi}$, $\phi \equiv \arg(z)$
and set $f''(t_0) = |f''(t_0)| \exp(i\phi_{f''})$
$\Rightarrow \mathcal{I}(z) \approx \int_C g(t) e^{zt} \, dt$ or
$\mathcal{I}(z) \approx g(t_0) e^{zt} \int_C \frac{d\tau}{\tau z} e^{zt} \frac{1}{2} |z| f''(t_0) \tau^2$
where we have chosen \( \alpha \) such that
\[
\frac{\pi}{e} (\phi_\pi + \phi_{\pi''}) + 2i\alpha = i\pi \\
\Rightarrow 2\alpha + \phi_\pi + \phi_{\pi''} = \pi
\]
or
\[
\alpha = \frac{\pi - \phi_\pi - \phi_{\pi''}}{2}
\]
or
\[
\alpha = \frac{\pi - \arg(z) - \arg(f''(t_0))}{2}
\]
Because the integrand now decays rapidly we change

\[
\int_0^\infty e^{-t} dt \rightarrow \int_\infty^{-\infty} e^{-t} dt
\]
and use the known integral
\[
\int_\infty^{-\infty} e^{-z^2} \, dz = (\frac{2\pi}{1A})^{1/2}
\]
to obtain finally
\[
\mathcal{I}(z) = q(t_0) e^{i\frac{\pi}{2}} e^{\frac{2\pi}{1z f''(t_0)}^{1/2}}
\]
\[
f'(t_0) = 0 \quad \alpha = \frac{\pi - \arg(z) - \arg(f''(t_0))}{2}
\]

Aside: To decide between \( \alpha \) and \( \alpha + \pi \), pick whichever gives an acute angle w.r.t. the original contour \( C' \).
\[ t' = z t, \quad dt' = z dt \]

\[ \Rightarrow \Gamma(z+1) = \frac{z}{z} \int_0^\infty e^{z(\ln t - t)} \, dt \]

so \[ f(t) = \ln t - t, \quad f'(t_0) = \frac{1}{t_0} - 1 = 0 \]

\[ \Rightarrow t_0 = 1, \quad f''(t_0) = - \frac{1}{t_0^2} = -1, \quad f(t_0) = -1 \]

whereby \[ \phi_{f''} = \arg(f''(t_0)) = \pi \]

\[ \Rightarrow \alpha = \frac{\pi - \phi_f - \phi_{f''}}{2} = 0 \]

\[ \Rightarrow \Gamma(z+1) \approx e^{-z} \cdot e^{-\frac{z}{2}} \left( \frac{2\pi}{1 - (-1)} \right)^{\frac{1}{2}} e^{-\frac{z}{2}} \]

or \[ \Gamma(z+1) \approx (2\pi)^{\frac{1}{2}} e^{-\frac{z}{2}} \sqrt{z + \frac{1}{2}} - \frac{z}{2} \]

(Stirling's formula)

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Some values

<table>
<thead>
<tr>
<th>z</th>
<th>( \Gamma(z+1) )</th>
<th>Stirling Approx</th>
<th>Fractional error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.922</td>
<td>-8 %</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>118.0</td>
<td>-1.7 %</td>
</tr>
<tr>
<td>10</td>
<td>( 3.63 \times 10^6 )</td>
<td>3.60 (\times 10^6 )</td>
<td>-0.8 %</td>
</tr>
<tr>
<td>15</td>
<td>( 1.31 \times 10^{12} )</td>
<td>1.30 (\times 10^{12} )</td>
<td>-0.6 %</td>
</tr>
</tbody>
</table>
Next, suppose \( z = |z| e^{i \phi_z} \) with \( \phi_z \neq 0 \)

Now \( \alpha = \frac{\pi - \phi_z - \pi}{2} = -\frac{\phi_z}{2} \)

and our asymptotic approximation gives

\[
\Gamma(z+1) = e^{-z} e^{\frac{-i \phi_z}{2}} - \frac{1}{2} z \left( 2\pi \right)^{\frac{1}{2}} z^{-\frac{1}{2}}
\]

But \( |z| e^{-i \phi_z/2} = z \)

So for complex \( z \), we still have the same formula that we derived for real, positive \( z \!:

\[
\Gamma(z+1) = \left( 2\pi \right)^{\frac{1}{2}} z^{\frac{1}{2}} e^{-z}
\]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \Gamma(z+1) )</th>
<th>Stirling Approx</th>
<th>Fractional error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
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<td>0.509 - 0.111i</td>
<td>8.7%</td>
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<tr>
<td>( 5i )</td>
<td>-0.0070 - 0.00136i</td>
<td>-0.00168 - 0.00139i</td>
<td>1.7%</td>
</tr>
<tr>
<td>( 10i )</td>
<td></td>
<td></td>
<td>0.8%</td>
</tr>
<tr>
<td>( 15i )</td>
<td></td>
<td></td>
<td>0.6%</td>
</tr>
</tbody>
</table>
The Rest of the Asymptotic Series

Consider \( J(z) = \int \frac{e^{zf(t)}}{e^{\frac{z}{2}} \sqrt{\pi \sqrt{z + f(t_0)}}} \, dt \). This is just the first term of an asymptotic series. To derive subsequent terms, e.g., for higher accuracy, consider a variable substitution:

Let \( f(t) = f(t_0) - w^2 \)

which defines a new variable \( w \) of integration, \( w = w(t) \). This is real because our path is the one of steepest descent. (Include the phase of \( z \) in the definition of \( f(t) \) now, so \( z = \text{real, positive} \).)

\[ J(z) \approx e^{zf(t_0)} \int e^{-\frac{z}{2} w^2} \left( f(t_0) - zw^2 \right) e^{\left( \frac{d}{dw} \right) dw} \]

Once we derive a series expansion for

\[ \frac{dt}{dw} = \sum_{n=0}^{\infty} a_n w^n \]

we can evaluate

\[ J(z) = e^{zf(t_0)} \sum_{n=0}^{\infty} \frac{(1 + (-1)^n) \Gamma\left(\frac{n+1}{2}\right)}{2^{n+1}} a_n \frac{1}{z^{n/2 + 1/2}} \]

using \( \int_{-\infty}^{\infty} e^{-\frac{z}{2} w^2} w^n \, dw = 0 \) for \( n = \text{odd} \).

or

\[ J(z) \approx e^{zf(t_0)} \left( -\frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k} \left[ \Gamma\left(\frac{2k+1}{2}\right) \right] a_{2k} \right) \]
\[ w(t) = \left( t - t_0 \right)^2 \left[ \frac{1}{2} f''(t_0) \right] \left[ 1 - \frac{1}{3} \frac{f^{(3)}(t_0)}{f'(t_0)} (t - t_0) - \frac{f''(t_0)}{4!} (t - t_0)^2 \right] \]

\[ w(t) = \sum_{i=1}^{\infty} A_i (t - t_0)^i + A_2 (t - t_0)^2 + A_3 (t - t_0)^3 + \ldots \]

whose inversion reads

\[ t - t_0 = b_1 w + b_2 w^2 + b_3 w^3 + \ldots \]

with

\[ b_1 = \frac{1}{A_1}, \quad b_2 = -\frac{A_2}{A_3}, \quad b_3 = \frac{+2A_2 - A_1 A_3}{A_1 A_3}, \quad \ldots \]

\[ b_4 = \frac{5A_1 A_2 A_3 - 5A_2^2 - A_1 A_4}{A_1 A_3}, \quad \ldots \]

\[ \frac{dt}{dw} = b_1 + 2b_2 w + 3b_3 w^2 + \ldots \]

\[ a_0 = b_1, \quad a_2 = 3b_3, \quad \ldots \]

\[ a_{2k} = \frac{(2k+1)!}{2k} b \]

\[ a_{2k+1} = \frac{(2k)!}{2k+1} b \]
(treat real, positive \& here)

E.g. for \(1/(z+1)\) \(f(t) = \ln t - t\)

which can be expanded about \(t_0 = 1\) as:

\[ (-2)(f(t) - f(t_0)) = \left(t - 1\right)^2 - \frac{2}{3} \left(t - 1\right)^3 + \frac{1}{2} \left(t - 1\right)^4 - \frac{2}{5} \left(t - 1\right)^5 \ldots \]

\[ = 2w^2(t) \]

whose inversion looks like (see Mathematica notebook)

\[ t - 1 = \sqrt{2}w + \frac{2}{3}w^2 + \frac{1}{9\sqrt{2}}w^3 - \frac{2}{135}w^4 + \frac{w^5}{540\sqrt{2}} \ldots \]

\[ \Rightarrow \quad \frac{dt}{dw} = \sqrt{2} + \frac{4}{3}w + \frac{1}{3\sqrt{2}}w^2 \quad \ldots \]

and doing the integrals gives (see Mathematica notebook for the rest)
\[ j_0(t) = \frac{1}{\pi} \left( \frac{t}{2} \right)^l \frac{1}{l!} \int_{-1}^{1} e^{-\frac{izt}{l}} (1-t^2)^{l} \, dt \]

and write the integrand as

\[ e^{izt + \frac{l}{2} \ln(1-t^2)} = e^{i \frac{l}{2} \ln(1-t^2)} \]

i.e., for any chosen \( \xi \), \( f(t) = it + \frac{l}{2} \ln(1-t^2) \)

and \( f'(\xi_0) = 0 = i + \frac{l}{2} \frac{1}{1-\xi_0^2} (-2\xi_0) \)

which gives 2 roots,

\[ t_1 = \frac{il}{2} - \left(1 - \frac{l^2}{2^2}\right)^{\frac{1}{2}} \]
\[ t_2 = \frac{il}{2} + \left(1 - \frac{l^2}{2^2}\right)^{\frac{1}{2}} \]

or at \( |t| \to \infty \), \( t_{1,2} = \frac{il}{2} \pm 1 \)

\[ a = \pi \frac{\phi_{\frac{1}{2}} - \phi_{-\frac{1}{2}}}{2} \]

\[ f''(t_{2,1}) = \left(\frac{\pi}{l} - \frac{l}{2}\right) + i \left(1 - \frac{l^2}{2^2}\right)^{\frac{1}{2}} \geq \frac{\pi}{l} + i \]

Now, \( \phi_{\frac{1}{2}}''(l_{2,1}) = 2\pi - \epsilon \), \( \phi_{\frac{1}{2}}''(l_{2,1}) = 0 + \epsilon \), \( \phi_{\frac{1}{2}}'' = 0 \)

so \[ a_{2} = \frac{\pi - 2\pi}{2} = -\frac{\pi}{2} \]
\[ a_{2} = \frac{\pi}{2} = \frac{\pi}{2} \]
\[ a_{2} = \frac{\pi - 0}{2} = \frac{\pi}{2} \]
To calculate $e^{z f(t_1)}$ and $e^{z f(t_2)}$, first expand $z f(t_i)$ at large $z$.

\[ z f(t_1) \sim -l - i z + l \ln(2i l) - l \ln z \quad \text{as} \quad z \to \infty \]

\[ z f(t_2) \sim -l + i z + l \ln(-2i l) - l \ln z \quad \text{as} \quad z \to \infty \]

so \( e^{z f(t_1)} = e^{l(2i l) - l - i z} \) and \( e^{z f(t_2)} = e^{l(-2i l) - l + i z} \). \( e^{-i \pi/2} = i = e^{i \pi/2} \).

Now put everything together:

\[ j\ell(z) \longrightarrow \left( \frac{1}{z} \left( \frac{a}{l} \right) \right) \sqrt{2\pi} \left\{ \frac{e^{-l(a/l)}}{l^{3/2}} \right\} \left\{ \frac{e^{-(a/l)} e^{-i a/l}}{l^{3/2}} \right\} \]

\[ \times \left\{ i e^{i z} - e^{-i\ell(l)} e^{i z} \right\} \]

\[ = \left( \frac{1}{2i} \sqrt{2\pi} / l! \right) e^{-a/l} \frac{l^{3/2}}{z} \left( e^{i z} - e^{-l\pi/2} - e^{-i\left(z - \frac{l\pi}{2}\right)} \right) \]

But \( e^{-l\ell + \frac{1}{2}} \sqrt{2\pi} = l! \)

\[ \Rightarrow \left\{ j\ell(z) \longrightarrow \frac{\sin(z - \frac{l\pi}{2})}{z} \right\} \]

\[ \text{as} \quad z \to \infty \]