e.g. #1

\[ S = 1 - 1 + 1 - 1 + 1 - \ldots \]

\[ S_n = 1, 0, 1, 0, 1, 0, \ldots \]

\[ \frac{S_n}{n} = 1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}, \frac{1}{2}, \ldots \]

\[ \lim_{n \to \infty} \frac{S_n}{n} = \frac{1}{2} \]

**does this make sense?**

Consider the binomial series for \( \frac{1}{1+x} \):

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots \]

\[
\Rightarrow 1 - 1 + 1 - 1 + 1 - \ldots = S
\]

\[
x \to 1
\]

But \[
\frac{1}{1+x} \xrightarrow{x \to 1} \frac{1}{1+1} = \frac{1}{2}
\]

\[
\Rightarrow \text{Cesaro sum gives a sensible answer here!}
\]

e.g. #2

look at the divergent Cauchy product considered earlier:

\[
\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n}}
\]

i.e. \( A^2 \) with \( A = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) = 0.605...

The Cesaro sum of the Cauchy product is

\[
\begin{array}{c|c}
\hline
n & S_n \\
1 & 1 \\
2 & 0.29 \\
3 & 0.69 \\
4 & 0.31 \\
5 & 0.52 \\
1000 & 0.3654
\end{array}
\]

\[
\Rightarrow \text{again, it works!}
\]
Series of functions

\[ s(x) = \sum_{n=1}^{\infty} u_n(x) = \lim_{n \to \infty} S_n(x) \]

where \( S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) \)

\[ \Rightarrow \text{The ideas of convergence discussed before now apply to each } x \text{ separately.} \]

The key question: Does the series converge uniformly?

**Def.** Consider an arbitrarily small \( \varepsilon > 0 \). If an integer \( N \) exists, independent of \( x \) in \([a, b]\), such that

\[ |s(x) - S_n(x)| < \varepsilon \quad \forall \, n > N \]

and for all \( x \) in \([a, b]\)

then the series is said to be **uniformly convergent** over \([a, b]\).
e.g. let \( s(x) = \sum_{n=1}^{\infty} \frac{x}{(n-1)x+1}(nx+1) \)

whose \( n^{th} \) partial sum is \( s_n(x) = \frac{nx}{nx+1} \rightarrow \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases} \)

\( \Rightarrow \) the series is discontinuous at \( x=0 \),

and does not converge uniformly over \([0,1]\).

Uniform convergence is a sufficient condition to permit the following:

\[ \int_a^b s(x) \, dx = \lim_{n \to \infty} \sum_{n=1}^{\infty} \left( \int_a^b u_n(x) \, dx \right) \]

Note - unit convergence is a sufficient condition to permit this, but it is not a necessary condition.

\[ \int_0^1 s_n(x) \, dx = x^{n+1} \bigg|_0^1 = \frac{1}{n+1} \to 0 \quad n \to \infty \]

This is not uniformly convergent, but consider

\[ \int_0^1 s_n(x) \, dx = x^{n+1} \bigg|_0^1 = \frac{1}{n+1} \to 0 \quad n \to \infty \]

which is the correct integral over \( s(x) \).

Here term-by-term integration works, but the sufficient condition is not obeyed.
2. Interchange of limit processes with summation
i.e., let \( s(x) = \sum_{n=1}^{\infty} u_n(x) \) uniformly on \([a, b]\) and consider an \( x \in [a, b] \)

Then \( \lim_{x \to x_0} \left( \sum_{n=1}^{\infty} u_n(x) \right) = \sum_{n=1}^{\infty} \left( \lim_{x \to x_0} u_n(x) \right) \)

Aside - The analogous theorem for differentiation is **NOT** correct.

i.e., it is **NOT TRUE** THAT

if \( s_n(x) \to s(x) \) uniformly on \([a, b]\) and if
the derivative of \( s_n(x) \) exists for each \( n \),
then \( s'(x) \) exists and moreover
\( s'(x) = \lim_{n \to \infty} s_n'(x) \) on \([a, b] \)

e.g. Suppose the \( n \)th partial sum is
\( s_n(x) = \frac{\sin nx}{\sqrt{n}} \) for all real \( x \).

Then \( \lim_{n \to \infty} s_n(x) = s(x) = 0 \), and this limit
is approached uniformly.

But
\( s_n'(x) = \sqrt{n} \cos nx - \frac{1}{2} \frac{\sin nx}{n^{3/2}} \)
and \( \lim_{n \to \infty} s_n'(x) \) does not exist!
But the following, weaker statement is true:

3. The derivative of the series sum $s(x)$ equals the sum of term-by-term derivatives, i.e.,
\[ \frac{ds(x)}{dx} = \sum_{n=1}^{\infty} \frac{du_n(x)}{dx}, \]
if $u_n(x)$ and $\frac{du_n(x)}{dx}$ are continuous on $[a,b]$, and if $\sum_{n=1}^{\infty} \frac{du_n(x)}{dx}$ is uniformly convergent on $[a,b]$.

**Tests for uniform convergence**

**Weierstrass M-Test**

If $0 \leq u_n(x) \leq M_n$ for $n = 1, 2, 3, \ldots$, in $a \leq x \leq b$,
and if $\sum M_n$ converges, then $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on $[a,b]$.
(This is a useful test for absolutely convergent series.)

2. **Abel's Test**

If $u_n(x) = a_n F_n(x)$, $a_n$ not necessarily positive, $a_n$ is convergent,
and $F_n(x)$ are bounded, $0 \leq F_n(x) \leq M$, for all $x$ in $[a,b]$,
and if $F_n(x)$ are monotonically decreasing versus $n$,
then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,b]$. 
Taylor series \[ f(x + a) = \left( \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n}{dx^n} \right) f(x) \]

which can be compared with \[ e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \]

implies that we can write the Taylor series in operator form as:

\[ f(x + a) = \left( e^{ad} \frac{d}{dx} \right) f(x) \]

**Multidimensional functions**

\[ F(\vec{x} + \vec{a}) = F(\vec{x}) + \vec{a} \cdot \nabla F(\vec{x}) + \frac{1}{2!} \left( \frac{\vec{a} \cdot \nabla}{\hbar} \right)^2 F(\vec{x}) \]

Or in terms of the quantal momentum operator \[ \vec{p} = -i \hbar \nabla \]

this can be written in terms of the finite translation operator as

\[ F(\vec{x} + \vec{a}) = \left( e^{i \frac{\vec{a} \cdot \vec{p}}{\hbar}} \right) F(\vec{x}) \]
The binomial series

\[ (1 + x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n \]

when \( m = \text{integer} \geq 0 \):

\[ (1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \ldots + x^m \]

This generalizes to an infinite Taylor series if \( m \neq 0, 1, 2, 3, \ldots \)

i.e., if \( m \neq \text{integer} \) or \( m = \text{integer} < 0 \)

though we must interpret

\[ m! = \Gamma(m+1) = \text{Gamma function} \]

i.e., defined by relations given in the book.

Here, just note that \( \Gamma(z) \) obeys:

\[
\begin{align*}
(i) & \quad \Gamma(z+1) = z \Gamma(z) \\
(ii) & \quad \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}
\end{align*}
\]

Useful relationships!

When does the binomial series converge?

Look at

\[ \frac{a_{n+1}}{a_n} = \left[ x^{n+1} \frac{m!}{(n+1)!} \right] \frac{(n+1)!}{(m-n-1)!} \]

\[ = x \frac{m-n}{n+1} \]

And

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|, \quad \text{so the binomial series converges absolutely if } |x| < 1 \]
Aside

Polynomial expansion, for integer $M$, $N$:

$$(a_1 + a_2 + \ldots + a_M)^N = \sum_{P} \frac{N!}{n_1! n_2! \ldots n_M!} a_1^{n_1} a_2^{n_2} \ldots a_M^{n_M}$$

where $P = (n_1, n_2, \ldots, n_M)$

$\sum$ means all permutations of nonnegative integers $n_i$ such that $\sum n_i = N$

e.g. $(a+b+c)^5$

$$= a^5 + 5(a^4b + ab^4 + a^4c + ac^4 + bc + b^4c)$$

$$+ 10(a^3b^2 + a^2b^3 + a^3c^2 + a^2c^3 + b^3c^2 + b^2c^3)$$

$$+ 20(a^3bc + ab^3c + abc^3)$$

$$+ 30(a^2b^2c + ab^2c^2 + a^2bc^2)$$

$$+ b^5 + c^5$$
Inversion of Power Series

Sometimes we know

\[ y(x) = a_0 + a_1 x + a_2 x^2 + \ldots \]

but we want instead to know \( x \) as a function of \( y \).

\[ \Rightarrow \text{define } x(y) = b_0 + b_1 (y - a_0) + b_2 (y - a_0)^2 + \ldots \]

Result: \( b_0 = 0 \), \( b_1 = \frac{1}{a_1} \), \( b_2 = -\frac{a_2}{a_1} \),

\[ b_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5} \]

Why? Equate \( x - b_0 = b_1 (y - a_0) + b_2 (y - a_0)^2 + b_3 (y - a_0)^3 + \ldots \)

plug in \( y - a_0 = a_1 x + a_2 x^2 + \ldots \) etc.

and equate coefficients of each power of \( x \).

i.e. \( x = b_1 [a_1 x + a_2 x^2 + a_3 x^3 + \ldots] \)

\[ + b_2 [a_1 x + a_2 x^2 + a_3 x^3 + \ldots]^2 \]

\[ + b_3 [a_1 x + a_2 x^2 + a_3 x^3 + \ldots]^3 + \ldots \]

\[ \Rightarrow a_1 b_1 = 1 \quad (\text{coff of } x^1) \Rightarrow b_1 = \frac{1}{a_1} \]

\[ b_1 a_2 + b_2 a_1^2 = 0 \Rightarrow b_2 = -\frac{b_1 a_2}{a_1^2} \]

\[ b_2 = -\frac{a_2}{a_1^2} \]

\[ b_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5} \]

\[ b_4 = -5a_2^3 + 5a_1 a_2 a_3 - a_1^2 a_4 \]

\[ b_4 = \frac{-5a_2^3 + 5a_1 a_2 a_3 - a_1^2 a_4}{a_1^7} \]